The Homicidal Chauffeur

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Theme

PURSUIT-EVASION differential games had their beginning in the pioneering work of Isaacs.[1] These differential games are a generalization of the optimal guidance problem, in the sense that guidance strategies are to be found for each of *two* players having conflicting objectives. The players' strategies, respectively, are found to minimize and to maximize a specific criterion,[2] which is often taken to be the time to termination, or "capture."

The second-order differential equations describing the dynamics of the homicidal chauffeur game include two parameters, which are the speed ratio and the ratio of capture-radius to pursuer's minimum-turn radius. It is the purpose of this paper to show that this apparently simple, physically motivated differential game can lead to novel and remarkably complex solutions. It is shown that the classical necessary conditions often fall far short of providing the complete solution to the problem. On the contrary, a variety of "exceptional" lines are found to occur in the state space of the game. These lines are not predicted by the theory and they are of fundamental importance, since they separate regions of differing optimal strategies for either or both of the players. Isaacs discovered the barrier, universal line (singular arc), equivocal line, and a pursuer's dispersal line in his study of the problem. Other types of lines are found to include new dispersal lines, switch lines, switch envelopes, focal lines, and safe-contact motion. The switch envelope and focal line phenomena are believed to be new to the general theory of differential games.

Contents

The "homicidal chauffeur" is one of the best known of the pursuit-evasion problems originated by Isaacs.[1] In this differential game, the chauffeur (or pursuer, P) attempts to run down or "capture" the slower but more agile pedestrian (or evader, E). The evader naturally wishes to avoid capture, or failing that, to maximize the capture time. Termination, or capture, occurs when the relative separation falls below a certain capture radius, which is known to both players. The speeds are constant, and P's control is his turn rate, the magnitude of which is bounded, corresponding to hard left or hard right turns. E's control is the direction in which he chooses to run, so that P's speed advantage is partially offset by E's maneuverability. Both players are assumed to know the relative position and the parameters in the game. In the normalized equations of relative motion, P has unit speed and unit maximum turn rate, so that E's relative position (x, y) changes according to

$$\dot{x} = -\sigma y + \gamma \sin\theta, \qquad \dot{y} = -1 + \sigma x + \gamma \cos\theta \tag{1}$$

Here, $\gamma < 1$ is E's speed, θ is E's velocity direction measured clockwise from P's velocity, and σ is P's turn rate, $|\sigma| \leq 1$. The unit of distance is then P's minimum-turn radius, and the dimensionless capture-radius is denoted by the symbol $\beta > 0$. The relative motion can also be expressed in polar coordinates

$$\dot{r} = -\cos\phi + \gamma\cos(\theta - \phi), \qquad \dot{\phi} = -\sigma + [\sin\phi + \gamma \sin(\theta - \phi)]/r \tag{2}$$

where the bearing ϕ is also measured clockwise from P's velocity.

For a given set of parameters (γ, β), the object of the study is to determine P's control σ, and E's control θ, as functions of the relative position (x, y). Under optimal play, the capture-time $V(x, y)$ must then satisfy the "main equation"

$$\min_{\sigma} \max_{\theta} (dV/dt) = \min_{\sigma} \max_{\theta} [V_x(-\sigma y + \gamma \sin\theta) + V_y(-1 + \sigma x + \gamma\cos\theta)] = -1 \tag{3}$$

where V_x and V_y are partial derivatives of the capture-time payoff. The optimal controls are then given in terms of these gradients as

$$\sigma = \operatorname{sgn} S \triangleq \operatorname{sgn}(V_x y - V_y x), \qquad \frac{\sin\theta}{V_x} = \frac{\cos\theta}{V_y} = \frac{1}{(V_x^2 + V_y^2)^{1/2}} \tag{4}$$

in any "regular" region of the relative space where the adjoint vector has continuous first and second partial derivatives. The adjoint equations are found to be

$$\dot{V}_x = -\sigma V_y, \qquad \dot{V}_y = \sigma V_x \tag{5}$$

which can be solved in terms of trigonometric functions, whenever $S \neq 0$, so that $\sigma = \pm 1$.

An explicit "solution" to the game requires the numerical specification of the parameters γ and β, and more than 20 qualitatively different forms of solution are found to be possible. As shown in Fig. 1, the parameter space can be broken into five major regions, which will be distinguished from one another in the following:

The results given by Isaacs correspond only to parameters in Regions I and IIc of this chart. When β is very small (below locus L_1), P has poor maneuverability, and the capture region is a finite curvilinear triangle just ahead of P. This region is bounded by the "barrier," a locus of infinite discontinuity in the time-to-go, which contacts the capture circle at the bearing $\phi_{up} = \cos^{-1}\gamma$, thus defining the "usable part" of the capture circle. The universal line (UL) or singular arc bisects the capture

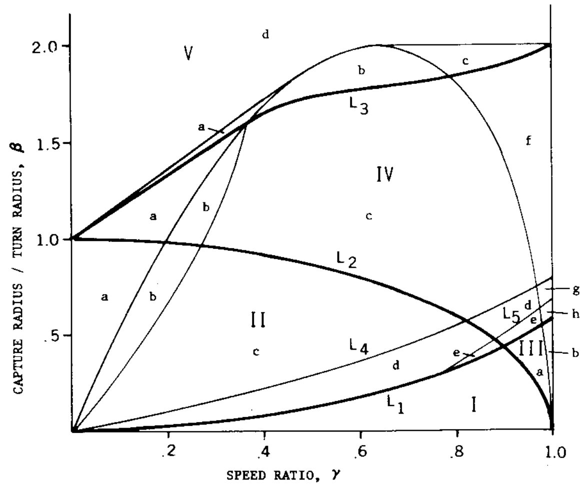

Fig. 1 Regions in parameter space.

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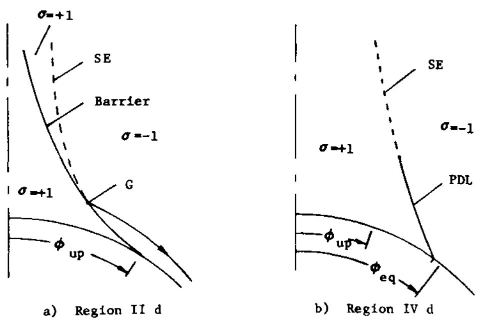

Fig. 2 Detail near switch envelope.

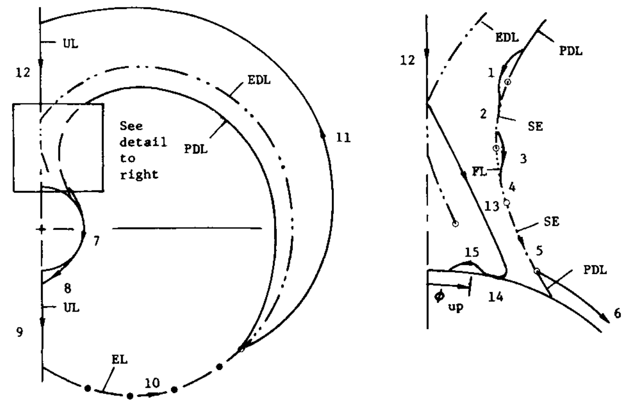

Fig. 3 A fifteen-stage game

region and this line corresponds to rectilinear motion of both P and E. When P is somewhat more maneuverable (β above L_1), the right and left barriers do not intersect ahead of P, but instead curve away to the rear. In such a case, capture is possible from all initial conditions, and the discontinuity in capture time at the barrier decreases to zero at its end. The barrier is then extended to the y-axis by the equivocal line (EL), which is a sharp minimum in the optimal time-to-go, $V(x, y)$; the normal to the local isochrone is, therefore, not defined on the EL, and E can choose between two strategies on this locus.

The parameters in Region III are associated with a finite capture region and a new barrier. This barrier terminates on the capture circle at an equilibrium point located at $\phi_{eq} = \sin^{-1}[(1+\beta^2-\gamma^2)/2\beta]$, and P and E describe concentric circular paths in realistic space in this equilibrium configuration. Located inside this barrier is either a pair of dispersal points (Region IIIa) or a pair of conjugate points (Region IIIb).

When the parameters are between L_1 and L_3, "safe-contact" trajectories are sometimes optimal, for which P turns *away* from E, who is initially located under the barrier but somewhat ahead of P. The relative trajectory meets and departs from the capture circle tangentially, and E follows a straight path in real space before and after the safe-contact portion. For all parameters above L_2 but below Region Vd, such safe-contact motion can also occur with P turning *toward* E. The common characteristic of these paths is $\dot{r} = 0$, which relates E's heading θ to the bearing according to Eq. (2).

For all parameters in Region IV, a curved pursuer's dispersal line exists on both sides of the y-axis. This PDL and a nearby EDL join at their far ends (contrary to the configuration shown in Ref. 3) and together they replace the barrier which exists for all parameters below L_2. The discontinuity in capture time which occurs at the barrier is thus replaced by a large but finite gradient. The barrier is extended to the negative y-axis by an ordinary switch line (SL) for parameters in Region II a, b. The PDL is similarly extended by an SL in Region IV a, b. When β is above the locus L_3, P never turns away from E, and the problem becomes relatively simple again.

The most complex games occur for parameters in the lower portion of Regions II and IV. Two new phenomena occur here, which are given the names "switch envelope" (SE) and "focal line" (FL). The SE occurs with (γ, β) in Region II when E is just under the barrier, which in this case does *not* coincide with a discontinuity in P's strategy. As illustrated in Fig. 2, when the retrograde solutions encounter the SE, both P and E switch strategies, the criterion for optimality being $V_x{}^+\dot{x}^- + V_y{}^+\dot{y}^- = -1$. In this equation,[3] the superscripts refer to before $(-)$ and after $(+)$ the SE is encountered, and a direct numerical test is required to find the point G of Fig. 2a. The area between the barrier and the SE is appropriately termed a "lunge" region for P, since, if E plays correctly, this is quickly followed by a "turn away" maneuver after the SE. The relative paths meet the SE tangentially, and, therefore, E must follow a curved path in realistic space if P chooses to continue the lunge maneuver after the SE is encountered. At the lowest point G of this region, P must finally turn away, but when E is above this point on the SE, P can turn sharply in either direction. The SE occurs for all parameters between L_1 and L_4 in Fig. 1. When the barrier does not exist, the SE extends a PDL, as in Fig. 2b.

The FL occurs for parameters between L_1 and L_5 in Fig. 1. The numerical generation of the SE shows that at some point, entering and exiting velocities are equal; i.e., with $\sigma = \pm 1$, E's corresponding controls θ^+ and θ^- are found by solving

$$\begin{aligned} \dot{x} &= -y+\gamma\sin\theta^+ = y+\gamma\sin\theta^-, \\ \dot{y} &= -1+x+\gamma\cos\theta^+ = -1-x+\gamma\cos\theta^- \end{aligned} \tag{6}$$

For motion along the FL, P can, therefore, choose to turn in either direction, and any number of switches are possible. In contrast, optimality permits only one switch on the SE. Both the SE and the FL have been found to occur in other pursuit-evasion games.[4,5]

Figure 3 shows the strategy regions and exceptional lines which correspond to parameters (γ, β) which are just above the locus L_1 in Region IVe. This is the most complex strategy configuration encountered in the study, and here an optimal solution can involve a 15-stage chase, which includes every qualitative feature of the game except the barrier and the SL. Further discussion of this example is given in Ref. 6, which describes in detail the computation of the loci of Fig. 1, and of the exceptional lines and pursuit-evasion strategies which correspond to each set of parameters.

References

[1] Isaacs, R., *Differential Games,* Wiley, New York, 1965, pp. 232–237, 278–286, and 297–304.

2 Bryson, A. E. and Ho, Y. C., *Applied Optimal Control,* Blaisdell Publishing Co., Waltham, Mass., 1969, Chap. 9.

[3] Breakwell, J. V. and Merz, A. W., "Toward a Complete Solution of the Homicidal Chauffeur Game," *Proceedings of the 1st International Conference on the Theory and Applications of Differential Games,* Amherst, Mass., 1969, pp. III-1–III-5.

[4] Breakwell, J. V., "Some Differential Games with Interesting Discontinuities," Navy Workshop on Differential Games, Aug. 3, 1973, U.S. Naval Academy, Annapolis, Md.

[5] Bernhard, P., "Linear Pursuit-Evasion Games and the Isotropic Rocket," SUDAAR No. 413, Ph.D. thesis, Dec. 1970, Stanford University Center for Systems Research, Stanford, Calif.

[6] Merz, A. W., "The Homicidal Chauffeur: A Differential Game," SUDAAR No. 418, Ph.D. thesis, March 1971, Stanford University Center for Systems Research, Stanford, Calif.